Square at Singular Cardinals

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February 23rd, 2013

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Logic in Southern California Meeting

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   - Introduction
   - Basics
   - Large Cardinals
2. Singular Cardinal Combinatorics
   - Motivation
   - Square and other principles
3. Results
   - First result
   - Second result
   - Future work

Introduction

- We study infinitary combinatorial principles at singular cardinals.
- Our goal is to narrow down the consistency strengths of these principles.
- Current upper bounds on consistency strengths are very high.
- We use the method of forcing combined with certain promising smaller large cardinals to lower these upper bounds greatly.

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A set α is an **ordinal number** iff it is transitive and well-ordered by $\in$. We will denote the proper class of all ordinals by On.

- Ordinal numbers are sets to which well-ordered linear orderings are order-isomorphic. They are commonly used for transfinite induction.
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Ordinal numbers
Construction

- We build ordinals inductively:
- $0 := \emptyset$
  - $\alpha + 1 := S(\alpha)$, where $S(x) = x \cup \{x\}$ is the *successor* function.
  - For $A = \langle \alpha_\xi \mid \xi < \gamma \rangle \subset On$, $\alpha = \bigcup A$ is an ordinal, and is the supremum/limit of A.
- Two types of ordinals:
  - α is a **successor** iff $\exists \beta : S(\beta) = \alpha$.
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  - $\aleph_0 = |\omega|$
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$$\mathrm{cf}(\alpha) = \text{the least limit ordinal } \beta \text{ such that there is an increasing}$$

$$\beta\text{-sequence } \langle \alpha_\xi \mid \xi < \beta \rangle \text{ with } \lim_{\xi \to \beta} \alpha_\xi = \alpha.$$

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Otherwise we say α is **singular**, that is, iff $\mathrm{cf}(\alpha) < \alpha$.

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- A set S is **closed** iff it contains all of its limit points. That is, the limit of any sequence bounded in S is contained in S.
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Large cardinals

- Large cardinals are infinite cardinals with whose existence is independent of ZFC.
  - They are **inaccessible** cardinals with various additional assumptions.
- We will use large cardinals in **consistency results**.
  - For example we will prove theorems of the form: Con(ZFC+LC) $\rightarrow$Con(ZFC+φ)
  - Shows that the **consistency strength** of φ is at most that of the large cardinal axiom LC.
  - For these proofs we build a model of set theory from an existing one. Often use **forcing** to build outer models.

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Elementary embeddings

Definition

A function $j : M \to N$ between models is called an **elementary embedding** iff it preserves first-order formulas.

- For example, if $j : M \to N$ is elementary, then

$$M \models (\forall z \in x)(z \cap \kappa = \emptyset) \longleftrightarrow N \models (\forall z \in j(x))(z \cap j(\kappa) = \emptyset)$$

- If κ is the least ordinal such that $j(\kappa) > \kappa$, then we call $\mathrm{crit}(j) = \kappa$ the **critical point** of j.

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Elementary embeddings & large cardinals

- Many large cardinals have both an elementary embedding definition and combinatorial definition.

Definition

If there exists a nonprincipal κ-complete ultrafilter on κ, we say κ is **measurable**.

Theorem

κ is measurable iff there is an elementary embedding $j : V \to M$ with $crit(j) = \kappa$.

- We will use both definitions as it is convenient.

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Motivation

- We are interested in how the universe behaves at (small) singular cardinals.
- Are there properties are independent of ZFC?f
- Numerous combinatorial principles which we may study.
  - Square, diamond, tree, stationary set reflection, etc.
- The existence of one type of combinatorial object might imply the existence or nonexistence of another.
- We look at the consistency strength of these principles holding or failing.

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Square principle

Definition

A sequence $\vec{C} = \langle C_\alpha \mid \alpha \in \lim(\kappa, \kappa^+) \rangle$ is called a $\square_\kappa$ sequence iff for each α

1. C_α is club in α
2. (Coherence) $\beta \in \lim(C_\alpha) \to C_\beta = \beta \cap C_\alpha$
3. $\mathrm{otp}(C_\alpha) \leq \kappa$

- We say "$\square_\kappa$ holds" iff there exists a $\square_\kappa$ sequence.

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Other square principles

- Common properties
  - Sequences of clubs or sets of clubs
  - Coherence

Examples

$\square_{\kappa,\gamma}$, $\square(\kappa^+)$, $\square^*_\kappa$, global square, etc.

- How can a square principle fail?
  - If every coherent sequence of squares has a **thread**, that is, a club that coheres with the sequence and reaches to the top.
  - We will often use large cardinals and their elementary embeddings to build threads.

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Definitions

A κ **tree** is a tree of height κ and width $< \kappa$.
We say that a cardinal κ has the **tree property** iff every κ tree has a cofinal branch.

- Observe that if $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ is a coherent sequence of clubs, then the following defines a tree ordering on κ:

$$\alpha <_T \beta \longleftrightarrow \alpha \in \lim(C_\beta)$$

Theorem

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(H.) It is consistent with ZFC and the existence of a quasicompact cardinal that* $\square(\aleph_{\omega+1}, <\omega)$ *fails.*

- Lowers the consistency strength of the failure of $\square(\kappa)$ at the successor to a small singular cardinal to QC^*.
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- Lowers the consistency strength of the failure of $\square(\kappa)$ at the successor to a small singular cardinal to QC^*.
- Here QC^* is a small strengthening of quasicompactness.
- Much lower than supercompactness, the existing upper bound.
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- Start with a large cardinal at which our desired property holds.
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Large cardinal notions

Definition

A cardinal κ is **subcompact** iff for every $B \subseteq \kappa^+$ there exists $\mu < \kappa$, $A \subseteq \mu^+$ and an elementary embedding

$$j : (H_{\mu^+}, \in, A) \to (H_{\kappa^+}, \in, B)$$

with $\text{crit}(j) = \mu$.

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Prikry forcing

- Forcing which turns a regular cardinal into a singular cardinal.
- Adds an ω-sequence cofinal in κ, thus singularizing κ.
- Requires κ to be measurable. Makes explicit use of the combinatorial definition.
- Nice properties include:
  - Preserves all cardinals
  - Does not add bounded subsets of κ.
  - Useful corollary - If $\mathrm{cof}(\gamma) < \kappa$ and we add a club C in γ, C must contain a club in γ in the ground model.

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- If we want to turn κ into $\aleph_\omega$, we must modify our Prikry forcing:
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A further result

Theorem

(H.) It is consistent with ZFC and the existence of a cardinal which is measurable and subcompact that κ is singular and $\square_{\kappa,2}$ holds while $\square_\kappa$ fails.

- This is similar to a more general result by Cummings, Foreman & Magidor at $\aleph_\omega$ which requires a supercompact.
- Purpose of this result is to separate out these square principles.
  - We have from ZFC that $\square_\kappa \to \square_{\kappa,2}$
  - (Schimmerling-Zeman) In models of the form $L[\vec{E}]$, κ not subcompact iff $\square_\kappa$ holds iff $\square_{\kappa,<\kappa}$ holds.
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Methods

- We do a preparation forcing involving:
  - A carefully chosen Easton support iteration
  - Our iteration add and then thread $\square_{\alpha,2}$ sequences for many $\alpha < \kappa$.
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Future work

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- Would like to generalize to

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